Abstract

These notes cover the fundamental theory of sequences in $\mathbb{R}$, including convergence criteria, boundedness, and the Bolzano–Weierstrass theorem.

Contents

1 Sequences in $\mathbb{R}$

A **sequence** in $\mathbb{R}$ is a function $a\colon \mathbb{N} \to \mathbb{R}$. We write $(a_n)_{n=1}^{\infty}$ or simply (a_n).

Recall that $\mathbb{N} = \{1, 2, 3, \ldots\}$ in this course.

A sequence (a_n) **converges** to a limit $L \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies |a_n - L| < \varepsilon.$$

We write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$.

Example 1.1. The sequence $a_n = 1/n$ converges to 0. Given $\varepsilon > 0$, choose $N > 1/\varepsilon$ (by the Archimedean property). Then for $n \geq N$:

$$|a_n - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

2 Boundedness and Monotonicity

Theorem 2.1 (Convergent sequences are bounded). *If $(a_n) \to L$, then (a_n) is bounded: there exists $M > 0$ such that $|a_n| \leq M$ for all n.*

Proof. Take $\varepsilon = 1$. There exists N such that $|a_n - L| < 1$ for all $n \geq N$. By the triangle inequality, $|a_n| < |L| + 1$ for $n \geq N$. Set

$$M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |L| + 1\}.$$

Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. □

Remark. The converse is false: the sequence $a_n = (-1)^n$ is bounded but divergent.

Theorem 2.2 (Monotone Convergence Theorem). *Every bounded monotone sequence in $\mathbb{R}$ converges.*

Proof. Suppose (a_n) is increasing and bounded above. Let $L = \sup\{a_n : n \in \mathbb{N}\}$, which exists by the completeness of $\mathbb{R}$. Given $\varepsilon > 0$, $L - \varepsilon$ is not an upper bound, so there exists N with $a_N > L - \varepsilon$. Since (a_n) is increasing, for $n \geq N$:

$$L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon.$$

Thus $|a_n - L| < \varepsilon$ for all $n \geq N$. □

3 The Bolzano–Weierstrass Theorem

Theorem 3.1 (Bolzano–Weierstrass). *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Proof sketch. Let (a_n) be bounded by $[-M, M]$. Apply the bisection method:

1. Set $I_1 = [-M, M]$.
2. At step k, bisect I_k. At least one half contains infinitely many terms; call it I_{k+1} and choose $a_{n_k} \in I_{k+1}$ with $n_k > n_{k-1}$.
3. The nested intervals I_k satisfy $|I_k| = 2M/2^{k-1} \to 0$, so by the Nested Intervals Theorem, $\bigcap_k I_k = \{L\}$ and $a_{n_k} \to L$. □

This proof relies on the completeness of $\mathbb{R}$. It fails in $\mathbb{Q}$.

Corollary 3.2. *A sequence in $\mathbb{R}$ converges if and only if it is a Cauchy sequence.*

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